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RR LYRAE STARS: A THEORETICAL STUDY OF BAILEY TYPES *a* AND *b*

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ABSTRACT

The full-amplitude behaviour of theoretical models of RR Lyr stars pulsating in the fundamental mode has been explored with the use of Carson's radiative opacities. A detailed comparison of the models with metal-poor RR Lyr stars of Bailey types *a* and *b* suggests that these stars have properties that lie somewhere in the following ranges: $M/M_{\odot} = 0.55$–0.65, $Y = 0.2$–0.3, and $\log (L/L_{\odot}) = 1.6$–1.7. In obtaining these results, the only data that are used are velocity curves and light curves. Independent information, however, seems to corroborate the values found here. The use of Los Alamos opacities gives somewhat similar results, although the derived masses are smaller by about 20%.

Subject headings: stars: interiors — stars: pulsation — stars: RR Lyrae

I. INTRODUCTION

Many authors have attempted to compute realistic nonlinear models of RR Lyr stars (Christy 1964, 1966*a*; Castor 1967; Baker and von Sengbusch 1969; Hill 1972; Spangenberg 1973; von Sengbusch 1973, 1975; Stellingwerf 1974, 1975; Hutchinson, Hill, and Lillie 1977; Deupree 1977*a*, *b*, *c*). Only Christy (1966*a*), however, seems to have tried to deduce masses and helium abundances of these stars by comparing theoretical light and velocity curves with those actually observed. Although he obtained good qualitative agreement with observations, the masses that he required, 0.35–0.55 $M_{\odot}$, now appear to be unacceptably small from an evolutionary point of view (see, e.g., van Albada and Baker 1971).

Perhaps the most influential, though still very uncertain, parameter governing the physics of the pulsating stellar models is the atomic opacity. Various versions of the Los Alamos opacities have been widely used by Christy and others, but the opacities that Carson (1976) has recently developed are significantly different from those produced at Los Alamos and have already led to considerably better agreement between theory and observation in the case of Cepheid variable stars (Vemury and Stothers 1978; Carson, Stothers, and Vemury 1981).

It appears to be of some interest, therefore, to calculate new models for RR Lyr stars with the use of the new Carson opacities. For the present paper, a set of full-amplitude models for Bailey-type *a* and *b* variables has been generated with these new opacities. The properties of the models will be discussed in § II and compared with observations in §§ III and IV. In addition, one model based on the most recent Los Alamos opacities will be presented.

II. NONLINEAR MODELS

It is our limited aim to present only a sufficient number of new models to permit an effective comparison with the models based on earlier opacities as well as with the basic observations of RR Lyr stars. Accordingly, we have chosen the following set of stellar parameters, guided mostly by the published work of Christy (1966*a*) and of Stellingwerf (1975): $M/M_{\odot} = 0.377$, 0.578, and 0.679; $L = 1.50 \times 10^{35}$ and 2.46×10^{35} ergs s^{-1}; and $T_e = 6200$, 6500, and 6800 K. For the helium and metals abundances we have adopted $Y = 0.25$ and $Z = 0.005$. In one case, we have used $(Y, Z) = (0.49, 0.02)$ in order to test the effect of increasing the helium abundance, the choice of Z being unimportant (Carson, Stothers, and Vemury 1981). Other physical assumptions and the computational techniques have been described by Vemury and Stothers (1978). In brief, the models have moderately deep stellar envelopes, radiation pressure in addition to gas pressure, no convection, Stellingwerf's (1975) version of the artificial viscosity, Christy's (1967) surface boundary conditions, and Carson's opacities for $\log T > 3.85$ (Carson, Stothers, and Vemury 1981). Only the fundamental mode of radial pulsation has been considered here.

Special notation in the paper includes: K. E., peak kinetic energy; Δ, full (not half) amplitude; *Asymmetry*, time spent on the descending branch of the surface velocity (or luminosity) curve divided by time spent on the ascending branch; ϕ_v, phase after minimum radius of the secondary bump on the surface velocity curve plus unity; $\delta\phi$ (max), phase of maximum luminosity minus phase of maximum velocity at the surface; and $\delta\phi$ (mean), phase of mean luminosity (defined by taking one-half of the sum of maximum and minimum bolometric magnitude) minus phase of mean velocity, during the main rise of luminosity at the surface. The term "surface" in our work refers to a mass layer of the equilibrium model where the optical depth is ~ 0.2.

Results for our full-amplitude models are shown in Table 1 and in Figures 1, 2, and 3. The luminosity and velocity curves in the first two figures have been smoothed slightly in the manner proposed by Carson

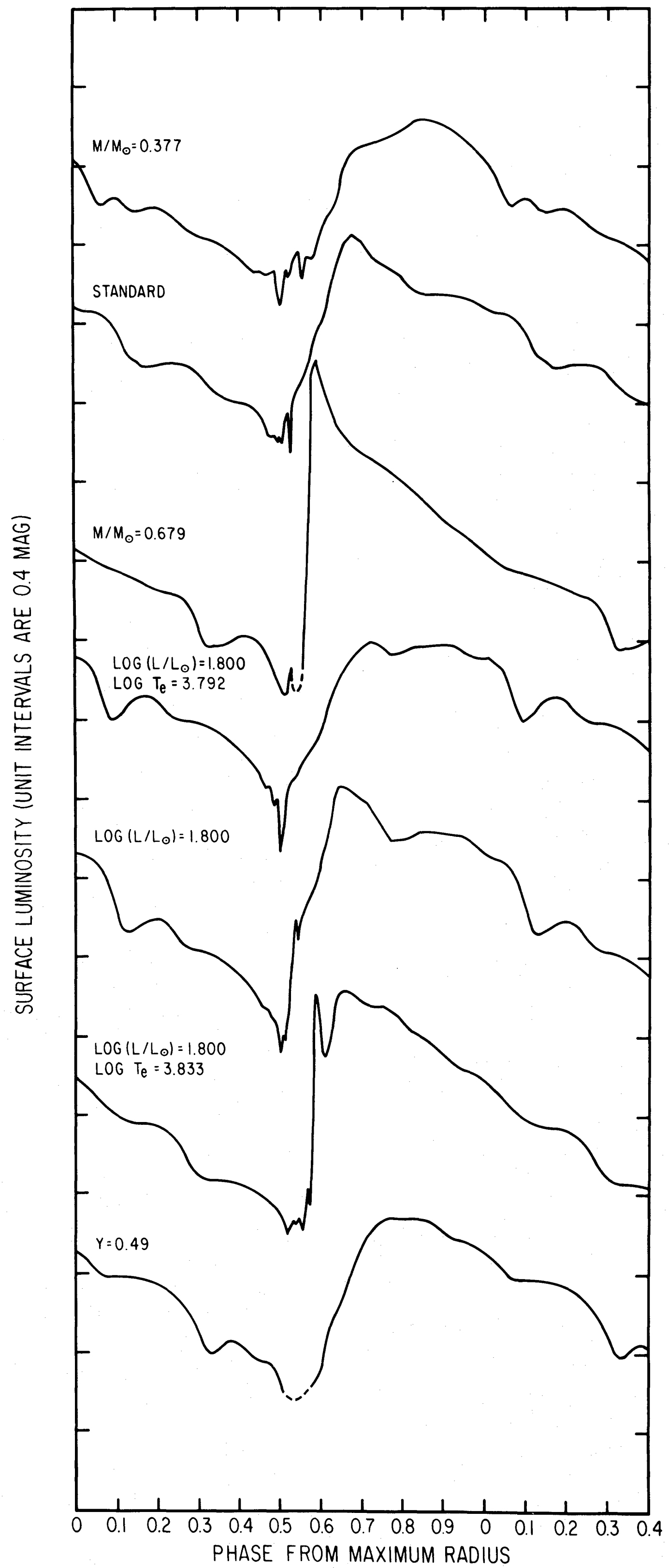

FIG 1.—Surface luminosity curves for the seven RR Lyr models. The "standard" model refers to $M/M_\odot = 0.578$, $Y = 0.25$, $\log (L/L_\odot) = 1.585$, and $\log T_e = 3.813$; the other models have parameter values that are different as indicated. Dashed segments indicate phases at which the computed luminosities are uncertain.

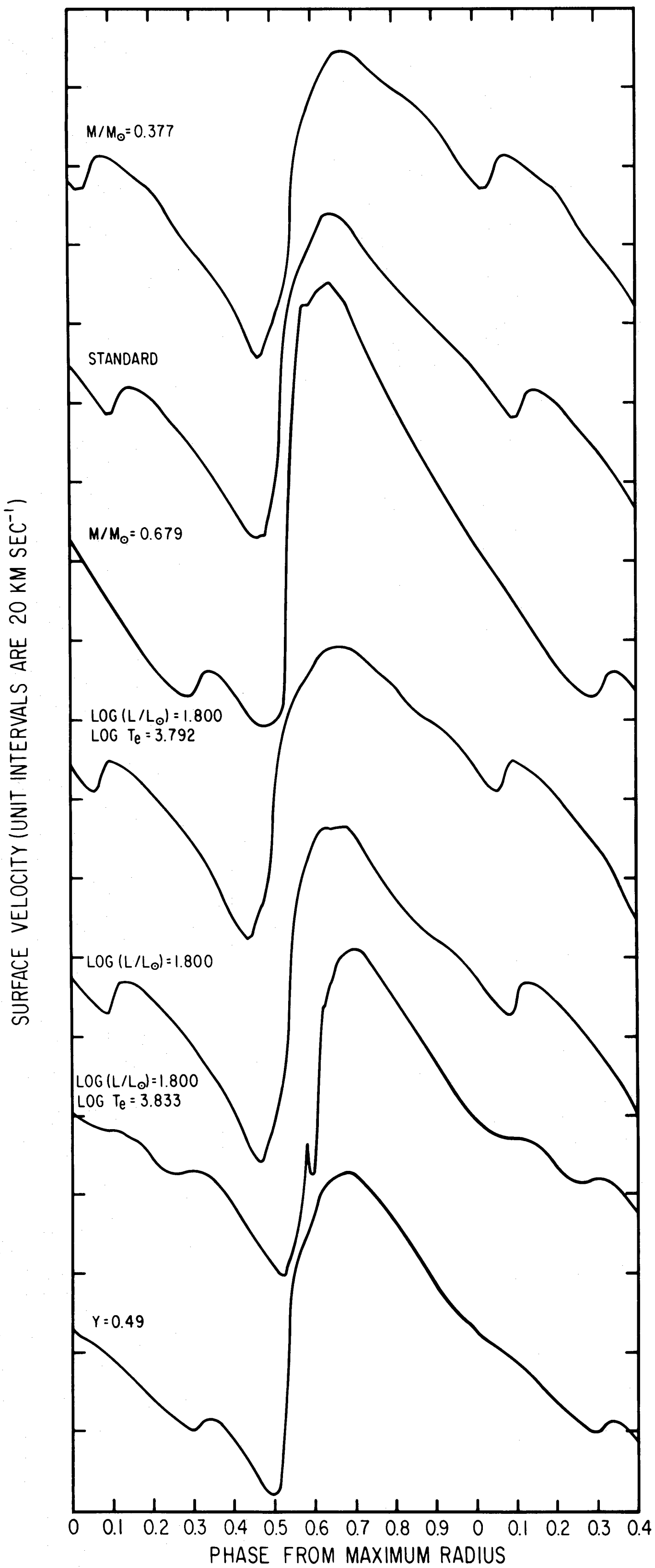

FIG. 2.—Surface velocity curves for the seven RR Lyr models. Same labeling as in Fig. 1

TABLE 1

FULL-AMPLITUDE PROPERTIES OF THE THEORETICAL MODELS OF RR LYRAE STARS

PARAMETER	MODEL 1	2	3	4	5	6	7
$M/M_{\odot}$	0.377	0.578	0.679	0.578	0.578	0.578	0.578
Y	0.25	0.25	0.25	0.25	0.25	0.25	0.49
$\log(L/L_{\odot})$	1.585	1.585	1.585	1.800	1.800	1.800	1.585
$\log T_e$	3.813	3.813	3.813	3.792	3.813	3.833	3.813
$R/R_{\odot}$	4.97	4.96	4.95	7.01	6.36	5.81	4.95
P (day)	0.732	0.529	0.487	0.984	0.823	0.702	0.542
K.E. (10^{40} ergs)	0.42	0.89	3.60	1.23	1.19	1.02	2.19
$\Delta R/R$	0.19	0.17	0.27	0.15	0.25	0.23	0.17
V_{out} (km s^{-1})	32	39	67	30	40	44	39
V_{in} (km s^{-1})	−46	−44	−47	−44	−47	−40	−41
ΔV (km s^{-1})	78	83	114	74	87	84	80
L_{max} (10^{35} ergs s^{-1})	2.1	2.3	3.4	3.2	4.0	4.1	2.1
L_{min} (10^{35} ergs s^{-1})	0.8	0.8	0.7	1.1	1.1	1.2	0.9
ΔM_{bol}	1.0	1.1	1.7	1.1	1.4	1.3	0.9
Asymmetry (vel.)	4.0	4.1	5.1	3.4	4.2:	4.9	4.2
Asymmetry (lum.)	2.4	5.7	17.2	3.5	6.1	9.0	3.3
ϕ_v	1.52	1.62	1.80	1.58	1.59	1.6±	1.81
$\delta\phi$ (max)	+0.18	+0.03	−0.05	+0.05	−0.02	−0.04	+0.09
$\delta\phi$ (mean)	+0.08	+0.06	+0.03	+0.07	+0.01	−0.03	+0.10

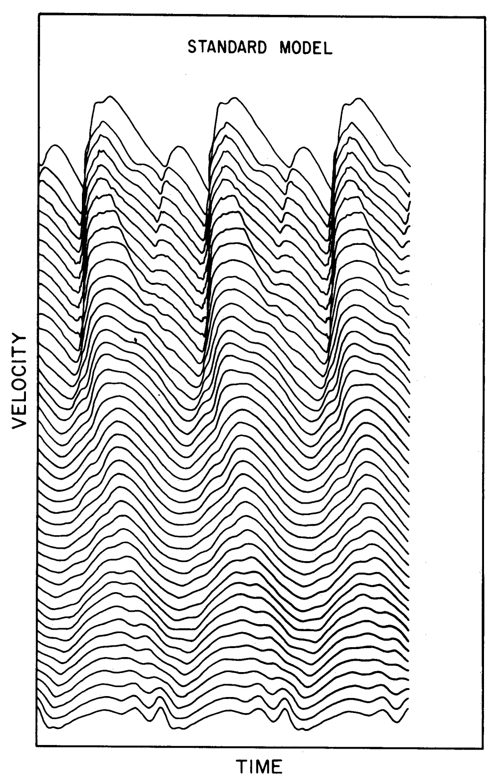

FIG. 3.—Velocity curves for the various zones in the "standard" RR Lyr model. The vertical scale is different for each zone.

et al. The model with $M/M_{\odot} = 0.578$, $\log(L/L_{\odot}) = 1.585$, $\log T_e = 3.813$, and $Y = 0.25$ shall be designated as our "standard" model; it is the approximate counterpart of Christy's (1966*a*) model 5gF.

Our new models differ from previously published models based on Los Alamos opacities in one very important respect: the phase ϕ_v of the small bump on the descending branch of the surface velocity curve in most of our models is approximately 1.6; in comparable earlier models it is about 1.9 (Christy 1966*a*; Baker and von Sengbusch 1969).[1] Stellingwerf (1975), however, did not obtain such a bump with the "Christy" version of the Los Alamos opacities, although he did find it near phase 1.5 with the "King Ia" version. On the other hand, he stressed that his results for the bump must be considered unreliable since the envelope depths in his models were very shallow. According to Christy (1966*a*, 1968), the phenomenon that produces the bump is a pressure wave that originates in the helium ionization zone and arrives at the surface only after a reflection from the stellar core; it may be traced in our Figure 3. It is evidently not caused by a resonance between the fundamental mode and one of the low overtones, because $P_1/P_0 \approx 0.74$ and $P_2/P_0 \approx 0.58$ in nearly all of our calculated RR Lyr models (§ IV). Its observable consequences can be seen also in the surface light curves; however, these curves are less reliably calculated in general than are the surface velocity curves.

[1] Note that the phase of the bump in Christy's (1966*a*) paper refers to the radial-velocity curve (in the observers' convention). This curve is the inverse of our velocity curve. Therefore what is visually perceived as a bump will be displaced slightly in phase between one curve and the other.

Except for the phase of the secondary bump, the rest of the features of our velocity and light curves resemble those computed by Christy (1966*a*), if we allow for the fact that the amplitudes of his light curves should be reduced somewhat because he ignored radiation pressure (see Baker and von Sengbusch 1969; Stellingwerf 1975). The bulk properties of our models, such as periods, are virtually the same as Christy's (1969*b*); we find, for example,

$$P_0 \approx 0.022(R/R_\odot)^{7/4}(M/M_\odot)^{-3/4} \text{ days },$$

with a scatter in the coefficient of only ± 0.0005.

To determine whether the most recent Los Alamos opacities yield significantly different results, we have recomputed our standard model with a new set of opacities kindly supplied by A. Cox. For $\log T < 4.1$ these opacities are the same as those published as equation (D3) in Stellingwerf (1975), although for $\log T \geq 4.1$ they are still in unpublished tabular form. Briefly, they seem to be rather similar to many earlier versions of the Los Alamos opacities and are shown, in part, in Figure 4. With respect to the Carson opacities (which are also shown, in part, in the same figure) they appear larger for $\log T < 4.6$, smaller for $\log T > 4.6$, and they have a less pronounced bump in the second helium ionization zone around $\log T = 4.6$, while entirely lacking a bump in the ultimate CNO ionization zone at $\log T > 5.3$ (which, however, has no importance for RR Lyr pulsations). By using these new Los Alamos opacities, our standard RR Lyr model is found to have the following characteristics: $P(\text{day}) = 0.536$; $\Delta R/R = 0.29$; $\Delta V(\text{km s}^{-1}) = 98$; $\Delta M_{\text{bol}} = 1.8$; Asymmetry (vel.) = 4.3; Asymmetry (lum.) = 19; and phase $\phi_v = 1.77$. Luminosity and velocity curves for this particular model are displayed in Figure 5. Compared with Christy's model 5gF, the new model develops larger amplitudes and an earlier phase of the secondary velocity bump, which was previously $\phi_v \approx 1.90$, according to Christy's (1966*a*) Figure 15, or $\phi_v \approx 1.85$, according to Baker and von Sengbusch's (1969) Figure 2. On the other hand, since the new surface amplitudes exceed those derived with the Carson opacities, it would appear that Carson's more prominent opacity bump in the second helium ionization zone does not drive the model to correspondingly larger amplitudes; in fact, as Christy (1966*a*) has already shown, a complicated saturation effect in this zone comes into play at large amplitudes. Only at small amplitudes, for example, near the blue edge of the instability strip, does the enhanced opacity bump have a simple and obvious effect (Carson, Stothers, and Vemury 1981).

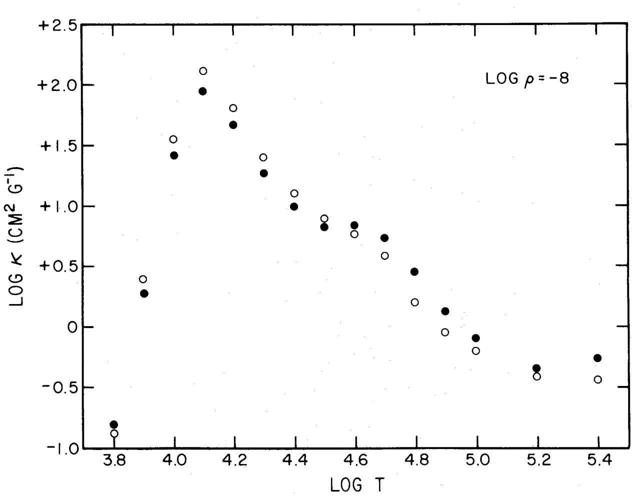

FIG. 4.—Stellar opacity as a function of temperature for $\rho = 10^{-8}$ g cm^{-3}: *filled circles*, Carson opacities; *open circles*, new Los Alamos opacities. Both sets of opacities are based on the same chemical composition, $(Y, Z) = (0.25, 0.005)$.

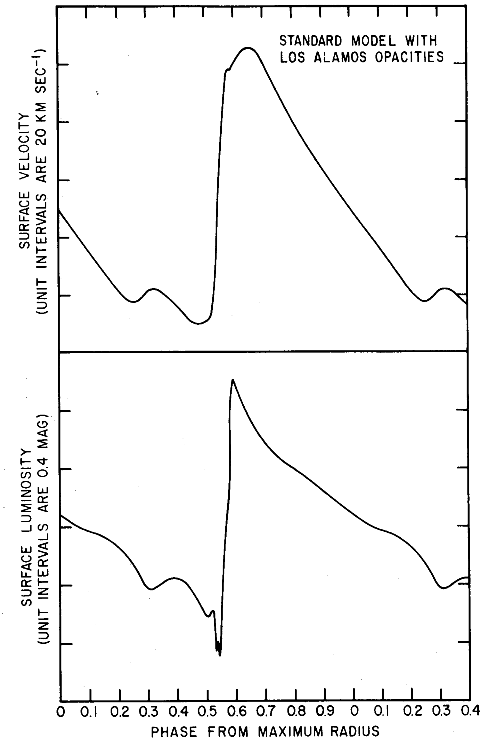

FIG. 5.—Surface velocity and surface luminosity curves for the "standard" RR Lyr model, recomputed with new Los Alamos opacities.

III. COMPARISON WITH OBSERVATIONS

Only four RR Lyr stars have been measured with sufficient accuracy and with sufficient phase coverage to discern a small bump, or inflection, on the radial-velocity curve shortly after the phase of maximum radius. (Sanford [1935] was the first to notice this bump.) These stars are XZ Cyg, RR Lyr, X Ari, and SU Dra—all metal-poor variables belonging to the general field, but resembling the variables found in globular clusters. Their pulsational properties are summarized in Table 2. The adjustment factor $\frac{24}{17}$ that multiplies the radial-velocity amplitudes represents an approximate correction for limb darkening and for geometrical projection (e.g., Wallerstein and Brugel 1979). In listing these velocity amplitudes, we have

TABLE 2

OBSERVED PROPERTIES OF FOUR RR LYRAE VARIABLES

PARAMETER	VARIABLE			
	XZ Cyg	RR Lyr	X Ari	SU Dra
P (day)	0.467	0.567	0.651	0.660
$\Delta R/R$	...	0.13	0.17	0.13
(24/17) ΔV_{rad} (km s^{-1})	85:	85	75	85
ΔM_{vis}	1.2	1.0	1.0	1.0
Asymmetry (vel.)	3:	4	4	6
Asymmetry (lum.)	6	6	6	6
ϕ_v	1.6:	1.6	1.6	1.6
References	3, 4	1, 2, 5, 6, 7, 14	9, 12, 15	8, 10, 11, 13

REFERENCES.—(1) Sanford 1935, 1949; (2) Struve and Blaauw 1948; (3) Struve and van Hoof 1949; (4) Muller 1953; (5) Hardie 1955; (6) Abt 1959; (7) Oke and Bonsack 1960; (8) Varsavsky 1960; (9) Preston 1961; (10) Spinrad 1961; (11) Oke, Giver, and Searle 1962; (12) Preston and Paczyński 1964; (13) Preston 1965; (14) Preston, Smak, and Paczyński 1965; (15) Oke 1966.

preferred radial velocities derived from metal lines to those derived from the hydrogen lines because the former give more consistent results (Oke, Giver, and Searle 1962) and because the "surface" in our stellar models refers to a fixed mass layer that has an average optical depth closer to the region where the metal lines are formed. Since XZ Cyg and RR Lyr show a noticeable Blazhko effect, i.e., a slow, but unexplained, modulation of the shape and amplitude of the observed light and velocity curves (possibly because of a magnetic field), the cycles of largest amplitude have been assumed, somewhat arbitrarily, to be most representative of the "normal" pulsation. This assumption is supported to some extent by the close similarity of the pulsational properties of the four stars when only the cycles of largest amplitude are adopted (see Table 2; also Preston, Smak, and Paczyński 1965). Note that the bolometric corrections of all these stars are very close to zero.

With regard to the basic pulsational characteristics, viz., period, amplitude, and asymmetry, these stars are mimicked well by our standard model computed with Carson's opacities. This model, it will be recalled, has $M/M_\odot \approx 0.6$, $\log (L/L_\odot) \approx 1.6$, and $Y = 0.25$. Even more interesting, however, is the agreement between the observed and computed values of ϕ_v, since this quantity has been found here, as it has been found before (Christy 1966*a*), to be a sensitive indicator of stellar mass. We find it also to be a good indicator of envelope helium abundance (although Christy found that it was not sensitive to the helium abundance with his version of the Los Alamos opacities). Our predicted values of ϕ_v, as well as our results for the light-curve asymmetry, can be used to reject, tentatively, masses as low as 0.4 $M_\odot$ or as high as 0.7 $M_\odot$, and helium abundances as high as $Y = 0.4$. Furthermore, if one accepts the predicted amplitudes, mean luminosities as bright as $\log (L/L_\odot) = 1.8$ can also be rejected.

The secondary bump on the light curve that corresponds to the one found on the velocity curve is observed at phase 0.65–0.75 after maximum light in the four stars of Table 2, as well as in many other RR Lyr stars (Lub 1977). From our models, this would suggest that the best-fitting mass lies in the range of 0.60–0.65 $M_\odot$, although, as discussed above, a slightly smaller mass seems to be indicated by the more reliably calculated (though less accurately observed) secondary bump on the velocity curve. It is important to stress that our results, unlike Christy's, require no independent knowledge of either the luminosities or effective temperatures of the observed stars. Absolute values of these quantities are, in fact, not known observationally with any great accuracy.

Still another unacceptable feature of the theoretical models with either very small mass or very high helium abundance is that they predict a significant lag of the light curve behind the velocity curve, amounting to 0.1–0.2 of a period. Observationally, it is known that maximum light always occurs at the phase of maximum outward velocity to within the error of measurement, which is ± 0.05 P. Similarly, mean light is known to coincide with mean velocity to within ± 0.01 P, at least over the period range $0.44 \leq P \leq 0.71$ days (Preston and Paczyński 1964). Unfortunately, our standard model is not perfect in this respect, because it shows $\delta\phi(\text{mean}) = +0.06$. One possible remedy would be to adopt a brighter luminosity, say, $\log (L/L_\odot) = 1.7$, which would lead to a zero phase lag when the period is about 0.55 days. To preserve the zero phase lag at both shorter and longer periods than this, it would be necessary to assume either that the masses of RR Lyr stars range by as much as a factor of 2 or that their mean luminosities and mean effective temperatures range in such a way that a mean brightness increase of as much as 0.4 mag is associated with a mean temperature drop of only 300 K. Both hypotheses clash noticeably with the accurate differential observational data available for the horizontal-branch stars in globular clusters (see, e.g., van Albada and Baker 1971).

Another possible remedy for the phase-lag problem would be to improve upon the simple radiative diffusion approximation used in the calculation of the models. However, a comparison of Christy's (1966*a*) radiative diffusion calculations for his model 5gF with Hill's (1972) more sophisticated radiative transfer calculations for the same model indicates that the incurred difference in phase lag is probably less than 0.02. This particular problem, which affects Christy's and Hill's models as it does ours, remains unsolved at present.

IV. OTHER MASS DETERMINATIONS

Christy (1966*a*) found an overall best fit of his models to the observations for an assumed average mass of $\sim 0.5\ M_\odot$. Since he employed observational data other than light and velocity curves in arriving at this conclusion, a more legitimate comparison ought to be based on the light and velocity curves alone. Christy himself provided a tentative comparison with three stars, relying principally on the observed phase of the secondary bump, and obtained 0.35–0.55 $M_\odot$ (see his § IV*b*, *d*). By using the most recent Los Alamos opacities, we can, in similar fashion, infer an average mass of $\sim 0.5\ M_\odot$. But in view of our derivation of a significantly larger mass with the Carson opacities, it is worth asking what other evidence exists concerning the masses of RR Lyr stars.

First, there is the uniquely determined mass for AQ Leo, a halo-population star that varies with two periods, $P_0 = 0.550$ and $P_1 = 0.410$ days; the ratio of the two periods indicates instability in the first overtone and fundamental mode (Jerzykiewicz and Wenzel 1977). Theoretical models of double-mode variables suggest that P_1/P_0 depends principally on P_0, mass, and chemical composition (Petersen 1978). To assess the difference made by the present opacities, we have computed periods for the models of Table 1 (and for two additional models having $Z = 0$ and $Z = 0.02$) by using linear nonadiabatic theory together with Baker-Kippenhahn photospheric boundary conditions. Our results can be summarized by an approximation formula:

$$P_1/P_0 = 0.772 - 0.037\ P_0 + 0.054 \log (M/M_\odot) + 0.03Y - 0.05Z^{1/3} ,$$

where P_0 is in days. Assuming $Y = 0.3$ and $Z = 0.001$, we find $M/M_\odot = 0.66$ for AQ Leo. By adopting Los Alamos opacities and the chemical composition just mentioned, Jerzykiewicz and Wenzel (1977) obtained $M/M_\odot = 0.70$ with the use of Stellingwerf's (1975) models, while Cox, King, and Hodson (1980) suggested $M/M_\odot = 0.65$ from their own models. We conclude that the results are insensitive to opacity. However, there is a mild dependence on Z: if $Z = 0$ (and $Y = 0.3$), then $M/M_\odot = 0.53$ with the Carson opacities.

Second, the masses of RR Lyr stars can be derived from the pulsational (P_0, M, R) relation. Although this relation is insensitive to both opacity and chemical composition, the empirical radii of RR Lyr stars are so uncertain that the deduced masses range all the way from 0.4 to 0.7 $M_\odot$ with mean errors of up to ± 0.2 (Oke, Giver, and Searle 1962; Oke 1966; Christy 1966*a*; Woolley and Savage 1971; van Albada and de Boer 1975; Woolley and Dean 1976).

A third method of estimating the masses employs measurements of the stars' surface gravities. But because these measurements need corrections for the large accelerations and because radii are again required, the derived masses show a similarly wide scatter from 0.4 to 0.6 $M_\odot$ with mean errors of $\pm 0.2\ M_\odot$ (Oke 1966; Jones 1973; McNamara and Feltz 1977). More reliable surface gravities can be obtained for blue horizontal-branch stars, which are believed to be in the same evolutionary stage as RR Lyr stars but to be slightly less massive; the masses of these blue stars average $0.54 \pm 0.16\ M_\odot$ (Newell, Rodgers, and Searle 1969*a*, *b*; Newell 1970) or, according to a more recent determination, $0.57 \pm 0.14\ M_\odot$ (Hayes and Philip 1979).

Finally, the location of RR Lyr stars in the H-R diagram, combined with theoretical evolutionary tracks, points to a range of masses, 0.55–0.80 $M_\odot$ (e.g., Iben and Rood 1970; van Albada and Baker 1971; Rood 1973; Sweigart and Gross 1976; Gingold 1976). It is only fair to mention that masses can also be derived from the location of certain transition lines in the H-R diagram for the fundamental mode and first overtone (e.g., Christy 1966*a*; Stellingwerf 1975), but the masses thus obtained have a large uncertainty and in any case cannot yet be compared with analogous masses based on the Carson opacities.

All in all, the independent evidence concerning the masses suggests an average mass somewhere between 0.55 and 0.65 $M_\odot$ for metal-poor RR Lyr stars.

V. CONCLUSION

A preliminary fit of our theoretical models based on the Carson opacities to the observed velocity curves and light curves of metal-poor RR Lyr stars of Bailey types *a* and *b* suggests that these stars have the following average characteristics: $M/M_\odot = 0.55$–0.65, $Y = 0.2$–0.3, and $\log (L/L_\odot) = 1.6$–1.7 (corresponding to a range of bolometric absolute magnitude, 0.4–0.7). Independent observational and theoretical data seem to support these derived values. They are also consistent with the values $M/M_\odot = 0.55$–0.60 and $Y \approx 0.3$ derived in similar fashion for short-period type II Cepheids, which are believed to be evolutionary derivatives, or at least close relatives, of RR Lyr stars (Carson, Stothers, and Vemury 1981). One drawback of the present models is that they predict a slight phase lag between the light and velocity curves that is not observed; the models also show no Blazhko effect.

Apart from these probably minor discrepancies, the models are rather successful on the whole, and lead one to suspect that the Carson opacities that have been used in the present work may be superior in some ways to previously adopted opacities. It must be remembered that the pulsational properties of variable stars in the Cepheid instability strip provide tests primarily of the hydrogen and helium opacities, which dominate the total opacity in the cool, outer layers where the pulsation amplitudes are significant. Therefore, it is interesting, though perhaps not surprising, that the derived bump masses of RR Lyr stars, type II Cepheids, and classical Cepheids all depend on a given set of opacities in roughly the same way. When the Los Alamos opacities are used, these masses are systematically smaller than when the Carson opacities are used.

Christy's (1966*a*) and our present work shows that, for RR Lyr stars, this mass difference amounts to 15–25%. On the other hand, his derived helium abundances and luminosities are similar to ours. But, since he had to use data (much of it not very reliable) other than just velocity and light curves to determine these two quantities, the agreement must be partly accidental. Nonetheless, his essential results have held up remarkably well.

It remains to extend the new survey of RR Lyr models to include the short-period (Bailey-type *c*) variables. Although some of these stars may be fundamental pulsators of low amplitude lying near the blue edge of the instability strip, the vast majority are probably pulsating in the first overtone. Christy (1966*a*) has shown that velocity curves and light curves for these stars are unlikely to reveal their masses and helium abundances. Therefore they do not properly belong to the present survey but will be taken up for investigation at a later date.

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